Study of shears in an artery affected by a stenosis or an aneurysm with the presence of two or three Isotropic invariants by using string rule approach

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Abstract:
In this research work we have proposed to study shears that can be observed in an artery affected by stenosis or aneurysm. A kinematic of transformation of those deseases was given and some tensors and isotropic invariants was calculated. we used the string rule to determine equations of shear derivative with an application on a compressible and isotropic model. In the one hand with the absence of one isotropic invariant, the exitential conditions and solutions of shear was given: five solutions were found where two are complex which are pure imaginaries and three are reals. In the other hand the generalization of the study mean with the presence of the three isotropic invariants gives only two complex solutions, existential condition is also given. The numerical simulation of real shear solutions shows difference between them and allowed us to find that there are two real solutions which better describes the shear in a stenosis or aneurysm artery.

Keywords:
Stenosis, Aneurysm, Elementary invariants, String rule, shear, compressible transformation, Isotropic energy potential.

1 Introduction

The study of shearing of elastic and incompressible materials has always been the subject of special attention in the study of mechanical systems [1]. In the study of mechanical fracture, the example of anti-planar shear has been a subject of particular interest to better understand these mechanical systems. Simple shear deformations, for which the displacement gradient is constant, are sustainable both in the linear and nonlinear theory. So that necessary and sufficient conditions on the strain energies for homogeneous isotropic nonlinear elastic materials which allow antiplane shear were obtained in Knowles for further contributions in the compressible case [2].
In the linear transversely isotropic elasticity, a study of deformation of a circular infinite hollow cylinder, whose inner surface is fixed, while its outer surface is subject to a constant axial surface traction is studied [3]. In isotropic linear

elasticity, the solution of this problem is just a state of anti-plane axial shear. The autors show that it is possible to use an axial tension field to generate an azimuthal shear deformation. they show that this fact suggests to use anisotropy to design some elastic machines which can combine different deformation modes. Other authors [4] have shown that this characterization of materials is closely related to the nature and form of the energy function. This characterization remains less obvious in nonlinear elasticity.
Other studies have focused on the effect of shear stress generated by a fluid in a tubular structure [5]. Their study showed that in the renal tube reduced fluid shear stress down-regulated the levels of megalin receptors, thereby reducing the renal distribution of albumin nanoparticles.
To describe the anisotropic hyperelastic mechanical behavior of a mechanical structure, it is still useful to use deformation energy functions in form polynomial, exponential, power or logarithmic. These energy potentials have been established as part of a phenomenological approach that describes the macroscopic nature of the material.
The study of shear in an artery affected by deseases like stenosis and aneurysm will be our contribution in the biomechanical modeling, we study a smooth hollow cylindrical structure subjected to shear in compressible and isotropic case by application to energy potential.
After the calculation of the different tensors and elementary invariants, we will use the string rule to determine the equations of shear derivative. The existential conditions and the solutions of shear will be given. Real shear solutions will be simulated and analyzed in order to highligth the difference between them.

2 Formulation of the problem

We consider a continuous material body. the whole of the particles of this body occupies, every moment, an open and connected domain or connected by arc of the physical space. The geometric domain is a hollow cylinder composed of an elastic, isotropic material with an inner surface bounded by a rigid cylinder and an outer surface subjected to axial shear. In a cylindrical coordinate system, let's consider a point M which in the undeformed configuration has the components (R, Θ, Z) and in the deformed configuration the components (r, θ, z). The kinematics of deformation representing the stenosis or euneurysm whith the presence of shear [6,7,8] is described by:

$$r(R,Z) = R \pm \delta \left(1 + cos\left(\frac{Z}{2}\right)\right) : -2\pi \leq Z \leq 2\pi; \quad \theta = \alpha_0 \Theta; \quad z = \gamma Z + \omega\left(R\right). \tag{1}$$

which translates for axial shear, a combined deformation of the tube: radial with $r(R)$ and longitudinal or anti-planar shear with $\omega(R)$, α_0 a parameter and γ is the elongation.

According to (1), we can calculate the deformation gradient tensor:

$$\mathbf{F} = \begin{pmatrix} r' & 0 & \lambda_{rz} \\ 0 & \alpha_0 \frac{r}{R} & 0 \\ \omega' & 0 & \gamma \end{pmatrix} \tag{2}$$

where r' and ω' are respectively the derivatives with respect to R of r and ω and λ_{rz} defined by $\lambda_{rz} = \partial r/\partial z$.
From the deformation gradient, we can calculate in the Eulerian configuration the left Cauchy-Green tensor defined by:

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \begin{pmatrix} 1+\lambda_{rz}^2 & 0 & \omega' + \gamma\lambda_{rz} \\ 0 & \left(\alpha_0 \frac{r}{R}\right)^2 & 0 \\ \omega' + \gamma\lambda_{rz} & 0 & \omega'^2 + \gamma^2 \end{pmatrix} \tag{3}$$

It's follow the adjoint of the tensor $\mathbf{B}$ noted $\mathbf{B}^*$ defined by $\mathbf{B}^* = det(\mathbf{B})\mathbf{B}^{-1}$, so then:

$$\mathbf{B}^* = \begin{pmatrix} \left(\alpha_0 \frac{r}{R}\right)^2 \left(\omega'^2 + \gamma^2\right) & 0 & \left(\alpha_0 \frac{r}{R}\right)^2 \left(\omega' + \gamma\lambda_{rz}\right) \\ 0 & \gamma^2 + \omega'\lambda_{rz}\left(\omega'\lambda_{rz} - 2\gamma\right) & 0 \\ \left(\alpha_0 \frac{r}{R}\right)^2 \left(\omega' + \gamma\lambda_{rz}\right) & 0 & \left(\alpha_0 \frac{r}{R}\right)^2 \left(1+\lambda_{rz}^2\right) \end{pmatrix} \tag{4}$$

With the two previous tenseur we can calculate the first three elementary isotropic invariants of deformation given by:

$$\begin{aligned} I_1 = tr(\mathbf{B}) &= 1+\lambda_{rz}^2 + \left(\alpha_0 \frac{r}{R}\right)^2 + \omega'^2 + \gamma^2; \\ I_2 = tr(\mathbf{B}^*) &= \left(\alpha_0 \frac{r}{R}\right)^2 \left(\omega'^2 + \gamma^2 + 1 + \lambda_{rz}^2\right) + \gamma^2 + \omega'\lambda_{rz}\left(\omega'\lambda_{rz} - 2\gamma\right); \\ I_3 = det(\mathbf{B}) &= \left(\alpha_0 \frac{r}{R}\right)^2 \left(\gamma^2 + \omega'\lambda_{rz}\left(\omega'\lambda_{rz} - 2\right)\right). \end{aligned} \tag{5}$$

Where tr defines the *trace* operator, det the *determinant* operator.

It should be noted that the elementary invariants allows us to obtain the energy potential W also called the deformation energy function which is a function of these invariants ($W(I_1, I_2, I_3)$). This function translates the mechanical and/or thermodynamic behavior of the arterial affected by stenosis.
Let's now choose as a potential that of Diouf-Zidi [9] which is sumarized in the compressible and isotrope case by:

$$W = \frac{\mu}{2}\left[(I_1 - 3) + a_1 (I_2 - 3) + a_2 \left(\left(I_3^{1/p} - 1\right)^p + (2-p)(I_3 - 1)\right) + a_3 \frac{2-p}{1+p} ln(I_3)\right] . \tag{6}$$

where μ, a_1, a_2, p and a_3 are material parameters.
For a specific choice of the parameter $p = 1$ and $a_3 = 0$, this previous potential becomes:

$$W = \frac{\mu}{2}\left[(I_1 - 3) + a_1 (I_2 - 3) + 2a_2 (I_3 - 1)\right] . \tag{7}$$

And then yields us these following partial derivatives:

$$\left\{ \; W_1 = \tfrac{\mu}{2}; \;\; W_2 = \tfrac{\mu a_1}{2}; \;\; W_3 = \mu a_2; \right. \tag{8}$$

where $W_{i(i=1,2;3)} = \partial W / \partial I_i$. With the remaining parameters, the respect of the Meredio conditions gives:

$$a_2 = -\left(a_1 + \frac{1}{2}\right). \tag{9}$$

3 Shears

In mathematical analysis, the derivative of composed functions by using the string rule is a very important tool for the determination of the derivatives of certain expressions which compose those functions.
In our study, the string rule applied on the partial derivatives of the energy potential is translated by the following relationships:

$$\begin{cases} \frac{\partial W}{\partial I_2} = \frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial r}\frac{\partial r}{\partial I_2}; \\ \frac{\partial W}{\partial I_2} = \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial r}\frac{\partial r}{\partial I_2}; \\ \frac{\partial W}{\partial I_1} = \frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial r}\frac{\partial r}{\partial I_1}. \end{cases} \tag{10}$$

And by using the fact that $\partial r / \partial I_i = (1/(\partial I_i / \partial r))$ the system (10) becomes:

$$\begin{cases} W_2 = \frac{W_1}{\omega'^2 + \gamma^2 + \lambda_{rz}^2 + 1}; \\ W_2 = W_3 \frac{\gamma^2 + \omega' \lambda_{rz}\left(\omega' \lambda_{rz} - 2\right)}{\omega'^2 + \gamma^2 + \lambda_{rz}^2 + 1}; \\ W_1 = W_3\left(\gamma^2 + \omega' \lambda_{rz}\left(\omega' \lambda_{rz} - 2\right)\right). \end{cases} \tag{11}$$

3.1 Particular solutions

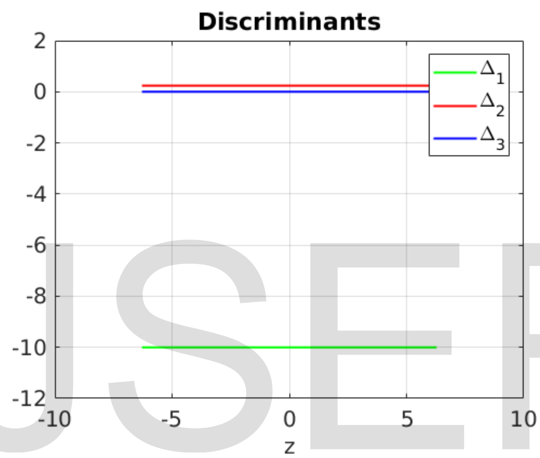
Let's remind that our objective of the study is to determine the solution of the shear. To illustrate the lack of one isotropic invariant, we will study this three previous equations separately. So by calculating this system, we obtain:

$$\begin{cases} \omega'^2 + \left(\gamma^2 + \lambda_{rz}^2 + 1 - \frac{W_1}{W_2}\right) = 0; \\ \left(W_2 - \gamma^2 \lambda_{rz}^2 W_3\right)\omega'^2 - 2\left(\gamma \lambda_{rz} W_3\right)\omega' + \left(W_2\left(\gamma^2 + \lambda_{rz}^2 + 1\right) - \gamma^3 W_3\right) = 0; \\ 4\left(\gamma \lambda_{rz} W_3\right)^2 \omega'^2 - 2\left(\gamma \lambda_{rz} W_3\right)\omega' + \left(\gamma W_3 - W_1\right) = 0. \end{cases} \tag{12}$$

We find three different second degree equations of ω'. To verify the nature of the solution in every equation, we consider that there is no variation of length i.e $\gamma = 1$, we note by $\varDelta_1$, $\varDelta_2$ and $\varDelta_3$ respectively the discriminant of $(12)_1$, $(12)_2$ and $(12)_3$, so we obtain:

$$\begin{cases} \varDelta_1 = -4\left(\gamma^2 + \lambda_{rz}^2 + 1 - \frac{W_1}{W_2}\right); \\ \varDelta_2 = 4\left[(\gamma\lambda_{rz}W_3)^2 - W_3\left(1-\gamma^2\lambda_{rz}^2\right)\left(W_2\left(\gamma^2+\lambda_{rz}^2+1\right)-\gamma^2 W_3\right)\right]; \\ \varDelta_3 = 4\left[(\gamma\lambda_{rz}W_3)^2 - (\gamma\lambda_{rz}W_3)^2\left(\gamma^2 W_3 - W_1\right)\right]. \end{cases} \tag{13}$$

A simulation of the three discriminants gives the following graphic.



From this previous graphic, we can see that we have real or complex solutions of the shear depending to the equation of the system (12). For the resolution of these equations, we will consider that there is no shear at the initial state, so:
The first equation of the system (12) gives two pure imaginary solutions which are:

$$\begin{cases} \omega_1(R) = -i\frac{\sqrt{\varDelta_1}}{2}R; \\ \omega_2(R) = i\frac{\sqrt{\varDelta_1}}{2}R; \end{cases} \tag{14}$$

where i is the complex number which verifies $i^2 = -1$.
The second equation of the system (12) gives two real solutions which are:

$$\begin{cases} \omega_3(R) = \frac{2(\gamma\lambda_{rz}W_3)-\sqrt{\varDelta_2}}{2(W_2-\gamma^2\lambda_{rz}^2W_3)}R; \\ \omega_4(R) = \frac{2(\gamma\lambda_{rz}W_3)+\sqrt{\varDelta_2}}{2(W_2-\gamma^2\lambda_{rz}^2W_3)}R. \end{cases} \tag{15}$$

The last equation of the system (12) gives a double solutions which is:

$$\omega_5(R) = \frac{R}{4\gamma\lambda_{rz}W_3}. \tag{16}$$

All the expressions of the shears show that we have two existential conditions which are:

$$\begin{cases} \lambda_{rz}^2 \neq \frac{a_1}{2\gamma a_2}; \\ \\ \lambda_{rz} \neq 0. \end{cases} \tag{17}$$

That gives us the existential conditions according to Z represented by:

$$\begin{cases} Z \neq 2\left(sin^{-1}\left(\pm\sqrt{\frac{a_1}{2\gamma a_2}}\right)\right); \\ \\ Z \neq 0, \quad Z \neq \pm 2\pi. \end{cases} \tag{18}$$

3.2 Proposal

Let f be a bijective function of reciprocal bijection denoted f^{-1}. If f is odd then its reciprocal bijection f^{-1} is also odd.

Proof

f a bijective function, so $f(x) = y \Longrightarrow f^{-1}(y) = x$.

$$\begin{aligned} f^{-1}(-y) &= f^{-1}(-f(x)) \\ &= f^{-1}(f(-x)) \\ &= f^{-1} \circ f(-x) \\ &= -x \\ &= -f^{-1}(y) \\ f^{-1}(-y) &= -f^{-1}(y) \end{aligned} \tag{19}$$

So f^{-1} is an odd function, hence the proof.

The previous proposal allowed us to have the final existential conditions which are:

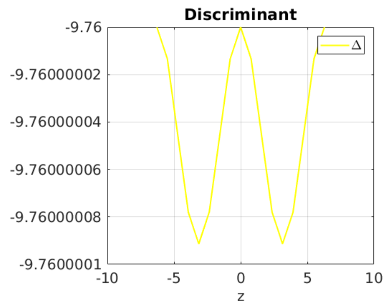
$$\begin{cases} Z \neq \pm 2\left(sin^{-1}\left(\sqrt{\frac{a_1}{2\gamma a_2}}\right)\right); \\ \\ Z \neq 0, \quad Z \neq \pm 2\pi. \end{cases} \tag{20}$$

3.3 General solution

Here we will carry the problem of shear of system (12) to one second degree equation by just making the sum of these three equations, we mean when the isotropic invariants are all present. That will gives us:

$$\left(1+W_2-\gamma^2\lambda_{rz}^2W_3+4\left(\gamma\lambda_{rz}W_3\right)^2\right)\omega'^2-4\left(\gamma\lambda_{rz}W_3\right)\omega'+\left(\gamma^2+\lambda_{rz}^2+1-\frac{W_1}{W_2}\right)$$
$$+\left(\left(W_2\left(\gamma^2+\lambda_{rz}^2+1\right)-\gamma^3W_3\right)+\left(\gamma W_3-W_1\right)\right)=0. \quad (21)$$

And when we denote by Δ the discriminant of the equation (21), we can obtain the following graphic:



A discriinant which is negative whatever the value of Z. And when we pose the following relationships:

$$\begin{cases} a=\left(1+W_2-\gamma^2\lambda_{rz}^2W_3+4\left(\gamma\lambda_{rz}W_3\right)^2\right); \\ b=-4\left(\gamma\lambda_{rz}W_3\right); \\ c=\left(\left(\gamma^2+\lambda_{rz}^2+1-\frac{W_1}{W_2}\right)+\left(W_2\left(\gamma^2+\lambda_{rz}^2+1\right)-\gamma^3W_3\right)+\left(\gamma W_3-W_1\right)\right), \end{cases} \quad (22)$$

with $\Delta=b^2-4ac$, the negative sign of the discriminant show that the general equation gives two complex solution which are:

$$\begin{cases} \Omega_1=\frac{-b-i\sqrt{\Delta}}{2a}R \\ \Omega_2=\frac{-b+i\sqrt{\Delta}}{2a}R. \end{cases} \quad (23)$$

The solutions of the generalization will exist if $a\neq 0$, what means:

$$sin^2\left(\frac{Z}{2}\right)\neq-\frac{4\left(1+W_1\right)}{\delta^2\gamma^2W_3\left(4W_3-1\right)}. \quad (24)$$

Proceeding as previously by using the proposal, the final relationship of the existential condition in the general case is:

$$Z\neq\pm 2sin^{-1}\left(\sqrt{-\frac{(4+2\mu)}{\delta^2\gamma^2\mu a_2\left(4\mu a_2-1\right)}}\right). \quad (25)$$

Let's specify that the term in the square root is always positive.

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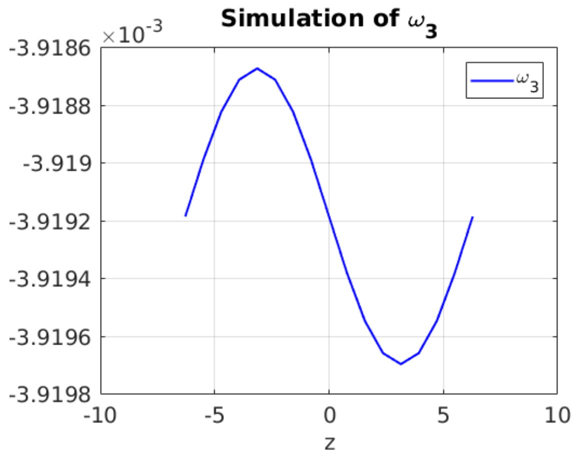
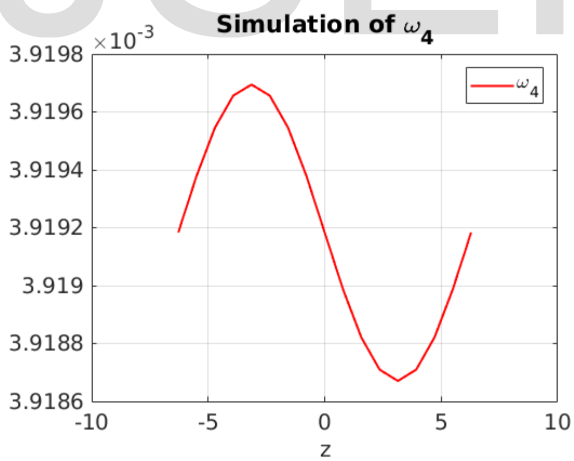
3.4 Real solutions simulation and interpretation

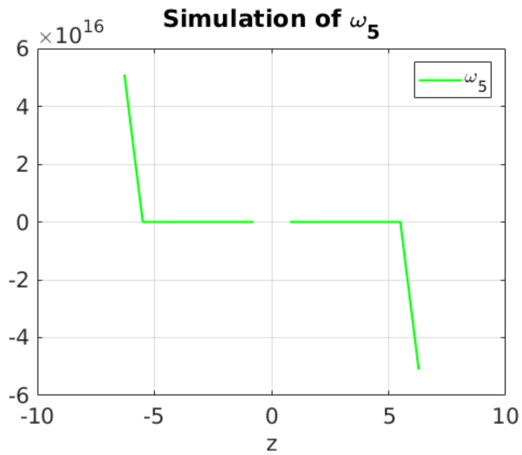
The simulation of the real shears with the chosen parameter at the level of the arterial radius shows the following graphics:



The solution of the shear ω_3 shows a behavior of a sinusoidal function according to the variation of Z. We also noted that the obtained values are all negatives. This behavior and values of ω_3 shows that this last is a good solution of the shear in the case of an artery affected by an aneurysm.



The solution of the shear ω_4 shows a behavior of a sinusoidal function also accordingto the variation of Z. For this time we noted that the obtained values are all positives.
This behavior and values of ω_4 shows that this last is a good solution of the shear in the case of an artery affected by an stenosis.



The solution of the shear ω_5 shows a behavior of constant function according to the variation of Z. We also noted that those obtained values are all zero excepted the two extreme values which carry the shear in $\pm\infty$.
This behavior and values of ω_5 shows that there is a sub interval of Z where this shear solution represents a neutral shear.

Remark

This study gives us five solutions of shear where two are pure imaginary expressions and three are real expressions when it done seperately from the equations of system (12).
The pure imaginary solution are always defined mean there always exist whatever the valu of Z.
Only the real solutions which are not defined in every valu of Z as we can see it in the system (20)

And when we sum the equation to generalize the study, we find two complex solutions of the shear problem. That means that in the genaral study, there is no real solution of the shear.

The simulation shows that the two distinct real solutions ω_3 and ω_4 of the shear can be superimposable, the only difference they have is that the first is negative and the second is positive.

4 Conclusion

In this study of shear of a hollow cylindrical tubular structure subjected to deformation representing an artery affected by stenosis or aneurysm, a kinematic of deformation describing those transformations obtained from the previous stud-

ies is given in the case of an isotropic and compressible model. Some tensor and elementary isotropic invariants needed in this study were calculated.
As our energy potential and invariants are composed function, we use the string rule to determine three equations of derivative of the shear which are all second degree equations. The resolution of those equations gives us five solutions of shear with three real solutions and two complex solution which are pure imaginaries when there are studied seperately. The existential conditions of shear was given. When the three equations are combined to yield a unique equation, only two complex solutions are founds and the existential condition is also given in this case.
The real shear solutions was simulated. The simulation shows some similitude of behavior between ω_3 and ω_4 and an infiny behavior of ω_5 at it boundaries. The interpretation of those simulated solutions shows that ω_3 can represent the shear in a artery affected by aneurysm and ω_4 that of an artery affected by stenosis in isotropic and compressible case.

5 Outlooks

As perpectives of our learning in biomechanic, this study can be a very important tool in the development and improuvement of surgical intervention in an artery affected by stenosis or aneurysm and even with the presence of fibrous reinforcement.

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